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ELECTRICAL CHARACTERISTICS OF THREE-COMPONENT DIELECTRIC COMPOSITES WITH CLOSE-PACKED INCLUSIONS

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The electric field and effective permittivity are calculated for a two-dimensional three-component dielectric material reinforced by cylindrical fibers. A composite material with a square close packing of inclusions is considered. The field in the periodic system is investigated using the exact solution of the model problem of interaction of two dissimilar cylindrical inclusions in an external homogeneous electric field. A diagram of the relative effective permittivity is constructed.

Introduction. In the present paper, we study properties of three- and multicomponent dielectric composites, in which polarization effects are more diverse compared to two-component dielectrics. Therefore, multicomponent dielectric materials have different properties.

Matrix composites with close-packed inclusions are considered. In such materials, even minor changes in the concentration and properties of individual components can have a significant effect on the characteristics of the heterogeneous medium as a whole. This makes it possible to produce sensitive gauges for measuring devices and to design materials with a wide range of characteristics for electron images and optical transducers. Among such materials are piezoelectrics, piezomagnetics, cermets, and thin film coatings in semiconducting and optical devices (see, e.g., [1, 2]).

In many cases, calculations of multicomponent systems with a high concentration of inclusions involve a number of mathematical difficulties, which prevent the development of general theory [3]. The difficulties that arise are due to the complication of the interaction of components in the system. As the distance between inclusions decreases, the shape of interacting bodies and their arrangement in the system and orientation in external fields acquires great importance. The interactions of bodies become multidipole, and parameters describing a heterogeneous system cannot be considered small. Therefore, in asymptotic expansions of solutions, it is necessary to retain a large number of terms.

The general theory of composites with periodic structure is described in detail [4]. The rigorous mathematical theory developed in this paper using asymptotic methods has made it possible to verify the validity of many model constructions, justify the hypothesis of equivalent homogeneity of periodically heterogeneous media, and estimate the errors of asymptotic calculations. Bakhvalov and Panasenko [4] showed that the calculation of average characteristics of piecewise-homogeneous periodic structures reduces to solving a boundary-value problem for a period cell. The latter is a complex problem and requires developing special mathematical methods of solution. Various approaches to solving this problem are described in [5, 6]. The class of inverse problems is of special interest.

The natural approach to determining averaged (effective) characteristics of a heterogeneous medium involves implementation of the following two operations: solution of a field problem and averaging of the calculated fields in a periodicity cell [7]. Usually, greatest difficulties arise in solution the first problem.

In the present paper, according to the general principles of theory, we determine the effective electrical characteristics of a dielectric medium containing inclusions in the form of unidirectional cylindrical fibers of two types. A three-component matrix medium is studied. Close-packed fibers form a periodic square lattice.

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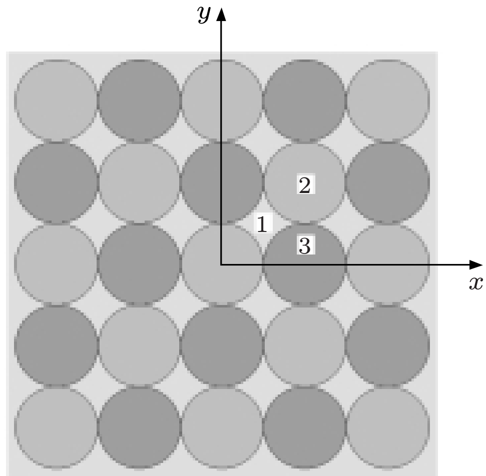

Fig. 1. Composite dielectric material with close-packed fibers of two types: matrix (1) and fiber with permittivities ε_2 (2) and ε_3 (3).

The material considered belongs to the class of orthotropic composites, whose properties are identical in the directions of two axes in a plane orthogonal to the fibers. The field equations admits rotation through a right angle around an axis whose direction coincides with the direction of the fibers.

To calculate the electric field in the system, we used a method based on summation of pair interactions. Use was made of solutions of two model problems of fibers in contact and two separate dissimilar dielectric fibers placed in an external homogeneous electric field. These problems have an exact analytical solution in a general formulation [8, 9].

At first glance, summation of multidipole interactions of a large number of inclusions leads to great complications in solution. In reality, the problem considered can be simplified: in a system with unbounded dimensions, it suffices to allow for interactions of an individual inclusion with its nearest neighbors only. This statement is valid for a dipole field, which, for plane systems, varies in inverse proportion to the square of distance. As a result, the mutual effect of inclusions diminishes with increase in distance between them. Long-range interactions can be ignored. Of course, the greater the heterogeneity of a medium, the larger the moments of interacting dipoles and, hence, the greater the radius of the region within which the interaction of the inclusions is taken into account.

The effective permittivity of multicomponent composite dielectrics satisfies the reciprocal relation, which can be written in several equivalent forms.

1. Electric Field in Inclusions. We study a dielectric medium with permittivity ε_1 which contains inclusions in the form of long parallel round fibers of radius r. Fibers of two types with permittivities ε_2 and ε_3 form a closed-packed square lattice. A fragment of a cross section a medium unbounded in space with doubly periodic alternation of inclusions of each type is shown in Fig. 1. The beginning of the rectangular coordinate system is on the axis of the inclusions with permittivity ε_2.

The system is in external homogeneous electric field E_0 directed normally to the axes of the cylindrical inclusions. In a cross section of the fibers, the electric field is two-dimensional, and in the absence of free charges, the basic equations of electrostatics $\nabla \cdot \boldsymbol{D} = 0$, $\nabla \times \boldsymbol{E} = 0$, and $\boldsymbol{D} = \varepsilon \boldsymbol{E}$ coincide with the Cauchy–Riemann conditions. Hence, we can introduce the complex quantities of induction and electric intensity $D(z) = D_x - iD_y$ and $E(z) = E_x - iE_y$ $(z = x + iy)$.

The electric field inside each inclusion is formed under the influence of the remaining inclusions of the system (interactions of an isolated inclusion with each other inclusion separately). Summing up these interactions, we obtain the resulting field in the isolated inclusion:

$$E_\nu(z) = \sum_{m=1}^{\infty} E_{\nu m}(z) \qquad (\nu = 2, 3). \tag{1}$$

Here the subscript ν corresponds to the isolated inclusion having permittivity ε_2 or ε_3 and $E_{\nu m}(z)$ is the electric-field component due to interaction of this inclusion ν with the inclusion m of the system.

Hence, to determine the electric field inside an inclusion, it is necessary to solve the problem of the electric field of two parallel dielectric fibers of cylindrical shape placed in an external homogeneous transverse field. This auxiliary problem is used in two versions: 1) dissimilar fibers of identical radius are in contact along the generator; 2) parallel fibers of circular cross section are separated by an arbitrary distance. The analytical solutions of these problems are different [8, 9].

With increase in distance between inclusions, the interaction between them weakens abruptly, which is valid for dipole systems in classical electrodynamics. Hence, we can take into account only interactions of an individual inclusion with its nearest neighbors and ignore its interaction with the remaining inclusions of the system.

Let, for example, we need to determine the electric field in an inclusion containing the coordinate origin (Fig. 1). In the adopted approximation, we can take into account only the effect of inclusions that are in the first layer surrounding the central inclusion. In this layer there are eight such inclusions: two pairs are in contact with the central inclusion (they are located on the coordinate axes) and the other two pairs of inclusions are on the diagonals $y = \pm x$, whose centers are separated from the coordinate origin by a distance of $2\sqrt{2}r$.

If higher calculation accuracy is required, we can take into account the effect of inclusions in the following layer, where they are already 16. Each subsequent layer has eight inclusions more than the previous one.

In what follows, the electric field and effective parameters are calculated under the assumption that the isolated inclusion interacts only with the nearest eight inclusions. As calculations showed, this assumption ensures satisfactory accuracy of results.

The adopted theoretical model makes it possible to convert from the infinite sum in expression (1) to a finite sum:

$$E_2(z) = \sum_{m=1}^{8} E_{2m}(z), \qquad |z| \leqslant r. \tag{2}$$

Here $E_2(z)$ is the electric intensity in the central inclusion, on whose axis the coordinate origin is located; the subscript 2 indicates that this inclusion has permittivity ε_2. The field components $E_{2m}(z)$ are obtained from the solution of the auxiliary problems.

For interaction with the inclusions on the positive and negative axes, we have

$$E_{21,22}(z) = E_0(1+\Delta_{12})\Big(\frac{1}{8} - \frac{1}{4}r^2 \sum_{k=1}^{\infty} \{\Delta^k k^{-2}[\alpha_k(z \mp a_k)^{-2} - (z \mp b_k)^{-2}]\}\Big), \tag{3}$$

where the upper sign (minus) and lower sign (plus) correspond to the positive and negative axes, respectively, $\alpha_k = (1/\Delta_{12})(2k/(2k-1))^2$, $a_k = 2kr/(2k-1)$, and $b_k = (2k+1)r/(2k)$ are the coordinate of dipoles on the x axis and $\Delta_{1\nu}$ and Δ are the relative parameters characterizing dielectric properties of the composite material:

$$\Delta_{1\nu} = (\varepsilon_1 - \varepsilon_\nu)/(\varepsilon_1 + \varepsilon_\nu) \quad (\nu = 2, 3), \quad \Delta = \Delta_{12}\Delta_{13} \quad (|\Delta_{1\nu}| \leqslant 1, \quad |\Delta| \leqslant 1). \tag{4}$$

In expression (3), the first term $E_0(1+\Delta_{12})/8$ defines the part of the homogeneous electric field in the central inclusion for one interacting inclusion (total of eight inclusions).

Generally, the medium is isotropic in cross section, and, therefore, the direction of the external field E_0 in the system can be chosen arbitrarily. For definiteness, the direction of the field E_0 is chosen along the x axis: $E_0 = E_{0x}$.

The presence of two other components of the field $E_2(z)$ in formula (2) is due to the interaction of the central inclusion with the adjacent inclusions located on the imaginary axis (Fig. 1). The expressions for them

$$E_{23,24}(z) = E_0(1+\Delta_{12})\Big(\frac{1}{8} - \frac{1}{4}r^2 \sum_{k=1}^{\infty} \{\Delta^k k^{-2}[\alpha_k(z \mp ia_k)^{-2} - (z \mp ib_k)^{-2}]\}\Big) \tag{5}$$

differ from expressions (3) only in the coordinates of the dipoles.

The remaining components of the field $E_2(z)$ result from the interaction of the central inclusion with the inclusions located on the diagonals $y = \pm x$ (Fig. 1) and are written as

$$E_{25,26}(z) = E_0(1+\Delta_{12})\Big(\frac{1}{8} - 4r^2 \sum_{k=1}^{\infty} \Big\{\Delta_{12}^{2k}\Big[\beta_k\Big(z \mp c_k \exp\Big(\frac{i\pi}{4}\Big)\Big)^{-2} - i\gamma_k\Big(z \mp d_k \exp\Big(\frac{i\pi}{4}\Big)\Big)^{-2}\Big]\Big\}\Big) \tag{6}$$

on the diagonal $y = x$ and

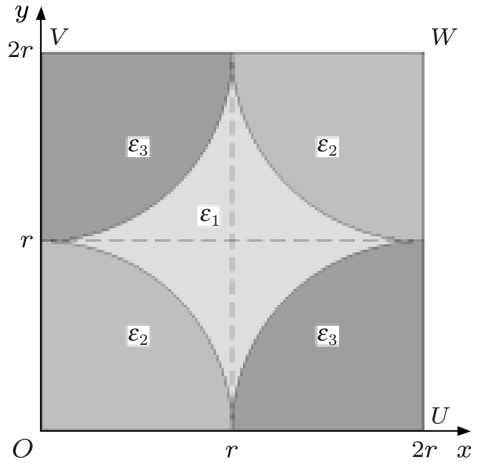

Fig. 2. Periodic cell of averaging of the electric field.

$$E_{27,28}(z) = E_0(1+\Delta_{12})\Big(\frac{1}{8} - 4r^2 \sum_{k=1}^{\infty} \Big\{\Delta_{12}^{2k}\Big[\beta_k\Big(z \mp c_k \exp\Big(-\frac{i\pi}{4}\Big)\Big)^{-2} + i\gamma_k\Big(z \mp d_k \exp\Big(-\frac{i\pi}{4}\Big)\Big)^{-2}\Big]\Big\}\Big) \tag{7}$$

on the diagonal $y = -x$. Here

$$\beta_k = \frac{1}{\Delta_{12}}\Big(\frac{\lambda^{k-1/2}}{1-\lambda^{2k-1}}\Big)^2, \quad \gamma_k = \Big(\frac{\lambda^k}{1-\lambda^{2k}}\Big)^2, \quad \lambda = 3 - 2\sqrt{2},$$

$c_k \exp(\,\cdot\,)$ and $d_k \exp(\,\cdot\,)$ are the coordinates of the dipoles on the diagonals $y = \pm x$:

$$c_k = \frac{r}{\sqrt{\lambda}}\,\frac{1-\lambda^{2k}}{1-\lambda^{2k-1}}, \qquad d_k = \frac{r}{\sqrt{\lambda}}\,\frac{1-\lambda^{2k+1}}{1-\lambda^{2k}}.$$

Expressions (3) and (5)–(7) define all terms in formula (2).

The electric field was calculated for an inclusion with permittivity ε_2. To determine the field $E_3(z)$ inside an inclusion with permittivity ε_3, we can use the obtained expressions, in which it is necessary to make replacement of the parameters $\Delta_{12} \leftrightarrow \Delta_{13}$. Thus, the coordinate origin moves to the center of the inclusion with permittivity ε_3.

In what follows, the expression for the electric field in the matrix medium $E_1(z)$ is not required, and, hence is not considered here.

In calculations using the method described, the error of the electric field can be judged by determining how rigorously the boundary conditions on the contours of the inclusions are satisfied. Direct check shows that in the adopted approximation, the condition of continuity the tangential components of the electric intensity and normal components of the induction vector on the lines separating the dissimilar media are satisfied with large accuracy.

2. Effective Characteristics. The effective permittivity ε_{eff} relates the average induction and average electric intensity:

$$\langle \boldsymbol{D} \rangle = \varepsilon_{\text{eff}} \langle \boldsymbol{E} \rangle. \tag{8}$$

Angular brackets in relation (8) denote volume-averaged magnitudes of the field. Averaging is performed over a space whose linear dimensions are larger than the characteristic dimensions of the volume element containing two types of inclusions. In this case, the average field is determined over the surface. By virtue of the periodic structure of the system, averaging is performed over a square cell which is an element of the period of the doubly periodic piecewise-homogeneous system considered (Fig. 2). Thus, determination of the effective permittivity of the material reduces to calculating relation (8) in a square cell.

By the condition of the problem, the external electric field E_0 is directed along the x axis. Therefore, in the cell OUWV, the direct segments OV and UW coincide with the equipotentials, and the segments OU and VW are field lines.

Averaging of the field over the surface area of the square cell OUWV can be replaced by calculation of contour integrals in this cell:

$$\langle E\rangle_x = \frac{1}{2r}\Big[\int_0^r E_{x2}(x)\,dx + \int_r^{2r} E_{x3}(x)\,dx\Big], \quad \langle D\rangle_x = \frac{1}{2r}\Big[\varepsilon_2\int_0^r E_{x2}(y)\,dy + \varepsilon_3\int_r^{2r} E_{x3}(y)\,dy\Big]. \tag{9}$$

Calculating integrals (9) from the relation $\langle D\rangle_x = \varepsilon_{\text{eff}}\langle E\rangle_x$, we obtain the required expression for the effective permittivity ε_{eff} of the examined material:

$$\varepsilon_{\text{eff}} = \varepsilon_1\,\frac{1 - s(\Delta_{12}+\Delta_{13}) + \Phi(\Delta_{21},\Delta_{31})}{1 + s(\Delta_{12}+\Delta_{13}) + \Phi(\Delta_{12},\Delta_{13})}. \tag{10}$$

Here $s = 0.3974$. If we increase the calculation accuracy, i.e., take into account the interaction of an inclusion with more than eight inclusions, the number s tends to the value of $\pi/8$, which is equal to the concentration of each type of inclusions in the material (in relative units). The difference between the approximate and exact values of the parameter s is about 1%.

Thus, the coefficient s in expression (10) has the physical meaning of the relative concentration for each type of inclusions. Therefore, below, we use the exact value of the parameter s equal to $\pi/8$.

According to relations (4), in expression (10) and subsequent formulas, we have

$$\Delta_{12} = -\Delta_{21}, \qquad \Delta_{13} = -\Delta_{31} \tag{11}$$

and, hence,

$$\Delta = \Delta_{12}\Delta_{13} = \Delta_{21}\Delta_{31}. \tag{12}$$

In expression (10), the functions $\Phi(\Delta_{12},\Delta_{13})$ are defined by

$$\Phi(\Delta_{12},\Delta_{13}) = \frac{1}{2}\sum_{k=1}^{\infty}\{\Delta^k[2(A_k+B_k) + (A_k+B_{k+1})(\Delta_{12}+\Delta_{13})]$$

$$+ (C_k+D_k)(\Delta_{12}^{2k}+\Delta_{13}^{2k}) + (C_k+D_{k+1})(\Delta_{12}^{2k+1}+\Delta_{13}^{2k+1})\}, \tag{13}$$

where

$$A_k = \frac{4(2k+1)^2}{(2k+1)^4-16k^4}, \quad B_k = \frac{4(2k-1)^2}{(2k-1)^4-16k^4}, \quad C_k = \frac{16\lambda^{2k+1}(1-\lambda^{2k+1})^2}{(1-\lambda^{2k+1})^4+\lambda^2(1-\lambda^{2k})^4},$$

$$D_k = \frac{16\lambda^{2k+1}(1-\lambda^{2k-1})^2}{(1-\lambda^{2k})^4+\lambda^2(1-\lambda^{2k-1})^4}, \qquad \lambda = 3-2\sqrt{2}. \tag{14}$$

If the permittivities of the inclusions ε_2 and ε_3 are close to the permittivity of the matrix ε_1 ($|\Delta_{12}|, |\Delta_{13}| \ll 1$), we can ignore terms of series (13) containing the parameters Δ_{12} and Δ_{13} to the second and higher power, i.e., we can set $\Phi(\Delta_{12},\Delta_{13}) = \Phi(\Delta_{21},\Delta_{31}) = 0$. The formula for the effective permittivity is thus simplified:

$$\varepsilon_{\text{eff}} = \varepsilon_1\,\frac{1-\pi(\Delta_{12}+\Delta_{13})/8}{1+\pi(\Delta_{12}+\Delta_{13})/8}. \tag{15}$$

Replacement of the linear-fractional function gives the linear expression

$$\varepsilon_{\text{eff}} = \varepsilon_1[1-(\pi/4)(\Delta_{12}+\Delta_{13})]. \tag{16}$$

Expression (10) defines the effective permittivity of an isotropic composite material with a regular close packing of cylindrical fibers of two types which form a square lattice. The relative concentrations of the two inclusions and the matrix are equal to 0.4, 0.4, and 0.2, respectively (more exactly, $\pi/8$, $\pi/8$, and $1-\pi/4$).

3. Analysis of Basic Formulas. We consider particular cases of the expression for the effective permittivity (10).

Two-Component Materials. 1. For $\varepsilon_2 = \varepsilon_3$ ($\Delta_{12} = \Delta_{13}$), we have a two-component composite with a close square packing of single-type fibers with concentrations of inclusions and matrix of 0.8 and 0.2 ($\pi/4$ and $1-\pi/4$), respectively. The effective permittivity of this material is given by

$$\varepsilon_{\text{eff}} = \varepsilon_1\,\frac{1-\pi\Delta_{12}/4+\Phi(\Delta_{21})}{1+\pi\Delta_{12}/4+\Phi(\Delta_{12})}. \tag{17}$$

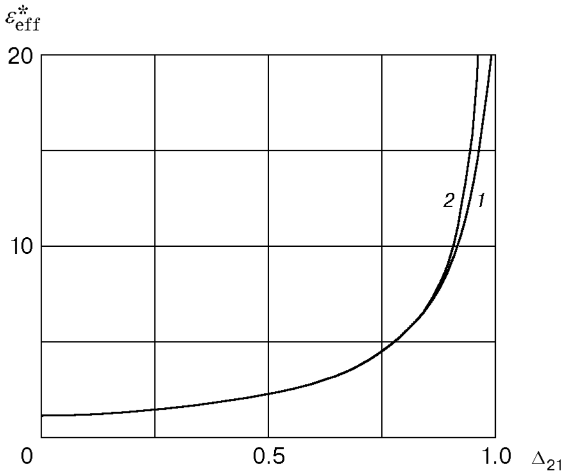

Fig. 3

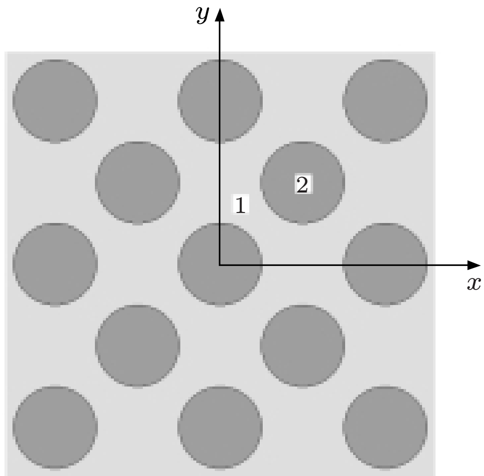

Fig. 4

Fig. 3. Effective permittivity of the two-component dielectric material versus the parameter Δ_{21}: curve 1 refers to calculation using the Rayleigh formula (18) and curve 2 refers to calculation by formula (17).

Fig. 4. Two-component dielectric material with periodic distribution of unidirectional fibers: 1) matrix; 2) fiber with dielectric permittivity ε_2.

Here, according to expression (13),

$$\Phi(\Delta_{12}) = \sum_{k=1}^{\infty}\{\Delta_{12}^{2k}[A_k + B_k + (A_k + B_{k+1})\Delta_{12} + C_k + D_k + (C_k + D_{k+1})\Delta_{12}]\},$$

where the coefficients at the parameter Δ_{12} are defined by formulas (14).

Studying a more general two-dimensional system [10] for the case considered, Rayleigh obtained the following expression (in the notation used in the present paper):

$$\varepsilon_{\rm eff} = \varepsilon_1 \frac{1 - \pi\Delta_{12}/4 - f_j(\Delta_{21})}{1 + \pi\Delta_{12}/4 - f_j(\Delta_{12})}. \tag{18}$$

The function $f_j(\Delta_{21})$ in expression (18) can be calculated with arbitrary accuracy (subscript $j = 1, 2, \ldots$ denotes the approximation order). For the third approximation, we have

$$f_3(\Delta_{12}) = -0.312\,\Delta_{12}^2 - 0.1594\,\Delta_{12}^3 + 0.0004\,\Delta_{12}^4. \tag{19}$$

The expressions for the effective permittivity (17) and (18) were obtained by different methods. Rayleigh employed classical potential theory. In the present paper, we use methods of the theory of functions of a complex variable. The relative dependences

$$\varepsilon^*_{\rm eff}(\Delta_{21}) = \varepsilon_{\rm eff}(\Delta_{21})/\varepsilon_1 \tag{20}$$

are presented in Fig. 3. In the range $0 \leqslant \Delta_{21} < 0.75$, the curves coincide, and for $\Delta_{21} > 0.75$ they differ only slightly.

Since formulas (17) and (18) are approximate, the question of which of them is more accurate for describing the effective permittivity for $\Delta_{21} > 0.75$ remains open. Theoretically, for $\Delta_{21} = 1$ ($\varepsilon_2 \to \infty$), the system considered should transform into a homogeneous (metal) state, and, hence, $\varepsilon^*_{\rm eff} \to \infty$. Actually, the parameter $\varepsilon^*_{\rm eff}$ calculated from formulas (17) and (18) takes values of 310 and 24, respectively. In this respect, in the adopted approximation, formula (17) is more accurate than formula (18). In a linear approximation for the parameter Δ_{21}, the analytical expressions of formulas (17) and (18) coincide.

2. If we set $\varepsilon_3 = \varepsilon_1$ ($\Delta_{13} = 0$) in the general formula (10), i.e., exclude one of the two components, we obtain a composite material with periodic arrangement of single-type fibers with concentrations of inclusions and matrix of 0.4 and 0.6 ($\pi/8$ and $1 - \pi/8$), respectively. A cross section of the structure of such a material is shown in Fig. 4.

Metal–Dielectric Transition. If two types of inclusions in the composite pass into the metal phase: $\varepsilon_2, \varepsilon_3 \to \infty$ ($\Delta_{12}, \Delta_{13} \to -1$), a metal–dielectric transition occurs in the system considered: $\varepsilon_{\rm eff} \to \infty$.

Characteristic Dielectric Medium. Let the permittivities of the matrix and inclusions be related by

$$\varepsilon_1 = \sqrt{\varepsilon_2 \varepsilon_3}, \tag{21}$$

which is equivalent to the equality

$$\Delta_{12} = -\Delta_{13}. \tag{22}$$

Then, taking into account relations (11) and (12), from expression (13), we obtain $\Phi(\Delta_{21}, \Delta_{31}) = \Phi(\Delta_{12}, \Delta_{13}) = \Phi(\Delta_{12})$, where $\Phi(\Delta_{12}) = \sum_{k=1}^{\infty}[(-A_k - B_k + C_k + D_k)\Delta_{12}^{2k}]$. The coefficients at the variable Δ_{12} are calculated from formulas (14). As a result, the expression for the effective permittivity (10) becomes

$$\varepsilon_{\text{eff}} = \varepsilon_1. \tag{23}$$

In the case where the permittivity of the matrix of a heterogeneous isotropic material is equal to the geometrical mean of the permittivities of the two types of inclusions, the effective permittivity of such a material is equal to the permittivity of the matrix, although the latter accounts for only about 20% of the material. From a physical viewpoint, this property of the effective permittivity is explained by the fact that in a heterogeneous dielectric for which relation (21) or (22) is satisfied, the permittivity of inclusions of one type is higher than the permittivity of the matrix, and the permittivity of the other type is lower. Therefore, the dielectric polarizations in dissimilar inclusions have opposite directions, and with rigorous satisfaction of equality (21), they are mutually cancelled. Since in a piecewise-homogeneous dielectric, polarization occurs on the matrix–inclusion boundary, it depends markedly on the boundary shape and not on the relative concentrations of the inclusions and the matrix.

A dielectric medium with the indicated properties is called a characteristic medium by analogy with the characteristic impedance of a transmission line, for which $Z = \sqrt{L/C}$, where Z is the resistance, L is the inductance, and C is the capacitance of the line elements).

4. Diagram of the Relative Effective Permittivity. For analysis of the averaged characteristics of a heterogeneous medium, it is convenient to introduce the relative effective permittivity:

$$\Delta_{\text{eff}} = \frac{\varepsilon_1 - \varepsilon_{\text{eff}}}{\varepsilon_1 + \varepsilon_{\text{eff}}}. \tag{24}$$

This quantity varies in the finite range of $-1 \leqslant \Delta_{\text{eff}} \leqslant 1$ for any permittivities of the matrix ε_1 and inclusions ε_2 and ε_3 or the parameters Δ_{12} and Δ_{13}. Substituting expression (10) into formula (24), we obtain

$$\Delta_{\text{eff}} = \frac{s(\Delta_{12} + \Delta_{13}) + \Phi^-(\Delta_{12}, \Delta_{13})}{1 + \Phi^+(\Delta_{12}, \Delta_{13})}. \tag{25}$$

Here $s = \pi/4$ is the total concentration of the two types of inclusions and the functions Φ^+ and Φ^- are defined by

$$\begin{aligned} \Phi^+(\Delta_{12}, \Delta_{13}) &= [\Phi(\Delta_{12}, \Delta_{13}) + \Phi(\Delta_{21}, \Delta_{31})]/2, \\ \Phi^-(\Delta_{12}, \Delta_{13}) &= [\Phi(\Delta_{12}, \Delta_{13}) - \Phi(\Delta_{21}, \Delta_{31})]/2. \end{aligned} \tag{26}$$

Using expression (13), from relations (26) we obtain

$$\begin{aligned} \Phi^+(\Delta_{12}, \Delta_{13}) &= \frac{1}{2}\sum_{k=1}^{\infty}[2(A_k + B_k)\Delta^k + (C_k + D_k)(\Delta_{12}^{2k} + \Delta_{13}^{2k})], \\ \Phi^-(\Delta_{12}, \Delta_{13}) &= \frac{1}{2}\sum_{k=1}^{\infty}[(A_k + B_{k+1})\Delta^k(\Delta_{12} + \Delta_{13}) + (C_k + D_{k+1})(\Delta_{12}^{2k+1} + \Delta_{13}^{2k+1})]. \end{aligned} \tag{27}$$

From expressions (27) it follows that the function $\Phi^+(\Delta_{12}, \Delta_{13})$ is even and the function $\Phi^-(\Delta_{12}, \Delta_{13})$ is odd with respect to the parameters Δ_{12} and Δ_{13}

If the permittivities of the inclusions ε_2 and ε_3 are close to the permittivity of the matrix ε_1 and, hence, the absolute values of the parameters Δ_{12} and Δ_{13} ($|\Delta_{12}|, |\Delta_{13}| \ll 1$) are small, then in expression (25) we can drop all terms containing the parameters Δ_{12} and Δ_{13} to the second and higher powers. This means that we can set

$$\Phi^+(\Delta_{12}, \Delta_{13}) = \Phi^-(\Delta_{12}, \Delta_{13}) = 0. \tag{28}$$

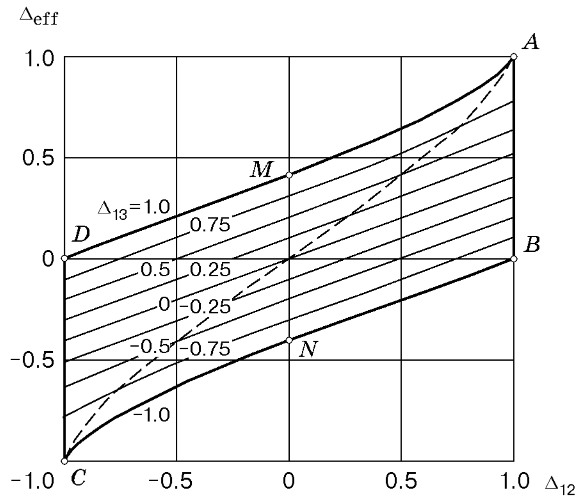

Fig. 5. Relative effective permittivity $\Delta_{\rm eff}$ versus the parameter Δ_{12} for various values of Δ_{13}.

With satisfaction of condition (28), the relative effective permittivity is written as a homogeneous linear function of the two variables:

$$\Delta_{\rm eff} = s(\Delta_{12} + \Delta_{13}). \tag{29}$$

This dependence can also be obtained from expressions (15) and (16).

Figure 5 shows the dependence $\Delta_{\rm eff}(\Delta_{12})$ for values of the parameter $\Delta_{13} = 0$, ± 0.25, ± 0.5, ± 0.75, and ± 1.0. The diagram gives the entire range of the effective permittivity of the dielectric medium studied. All values of the parameter $\Delta_{\rm eff}$ are bounded by the curvilinear tetragon ABCD. Several typical dependences and separate points can be distinguished on the diagram.

The dependence $\Delta_{\rm eff}(\Delta_{12})$ at $\Delta_{13} = 0$ ($\varepsilon_1 = \varepsilon_3$) corresponds to a binary system with hexagonal structure (see Fig. 4). The inclusions have permittivity ε_2. The same structure of material takes place for values of the parameter $\Delta_{\rm eff}$ on the segment MN: $\Delta_{12} = 0$ ($\varepsilon_1 = \varepsilon_2$) but the inclusions have permittivity ε_3.

On the segment DB, where $\Delta_{\rm eff} = 0$ ($\varepsilon_{\rm eff} = \varepsilon_1$), the composite material has the properties of a characteristic medium (see Sec. 3).

The boundary curves of the curvilinear tetragon ABCD describe properties of heterogeneous materials for which the permittivity of one of the two types of inclusions far exceeds the permittivity of the matrix or is well below it. Thus, the condition $\varepsilon_3 \gg \varepsilon_1$ ($\Delta_{13} = -1$) is satisfied on the curve AD, and the condition $\varepsilon_3 \ll \varepsilon_1$ ($\Delta_{13} = 1$) is satisfied on the curve BC; the relation $\varepsilon_2 \ll \varepsilon_1$ ($\Delta_{12} = 1$) is valid on the segment AB, and the relation $\varepsilon_2 \gg \varepsilon_1$ ($\Delta_{12} = -1$) is valid on the segment DC.

The points A and C correspond to the state of the material in which metal–dielectric transitions occurs: $\Delta_{\rm eff} = \pm 1$. From Fig. 5 it follows that a heterogeneous dielectric can pass into a homogeneous (metal) state under two conditions: 1) passage of both inclusions to the metal phase: $\varepsilon_2, \varepsilon_3 \to \infty$ ($\Delta_{12}, \Delta_{13} \to -1$) (point C in Fig. 5); 2) passage of the matrix material to the metal phase: $\varepsilon_1 \to \infty$ ($\Delta_{12}, \Delta_{13} \to 1$) (point A in a Fig. 5).

The coordinate origin in Fig. 5 corresponds to a homogeneous material. In its neighborhood, the effective characteristics of the composite $\Delta_{\rm eff}$ and $\varepsilon_{\rm eff}$ have the form of linear dependences on the parameters Δ_{12} and Δ_{13}, which also follows from expression (29).

The dashed curve in Fig. 5 corresponds to the dependence $\Delta_{\rm eff}(\Delta_{12})$ for a two-component material in which all inclusions in the close-packed system have identical permittivities: $\varepsilon_2 = \varepsilon_3$ ($\Delta_{12} = \Delta_{13}$).

5. Reciprocal Relations. Using expression (10), we can establish that the effective permittivity of an isotropic three-component material considered satisfies the reciprocal relation

$$\varepsilon_{\rm eff}(\Delta_{12}, \Delta_{13})\varepsilon_{\rm eff}(\Delta_{21}, \Delta_{31}) = \varepsilon_1^2, \tag{30}$$

where, by definition (11), replacement of the subscripts of the parameters Δ_{12} and Δ_{13} results in change of their signs: $\Delta_{12} = -\Delta_{21}$ and $\Delta_{13} = -\Delta_{31}$.

For a two-dimensional system [$\varepsilon_3 = \varepsilon_2$ ($\Delta_{13} = \Delta_{12}$)], the reciprocal relation (30) becomes

$$\varepsilon_{\rm eff}(\Delta_{12})\varepsilon_{\rm eff}(\Delta_{21}) = \varepsilon_1^2. \tag{31}$$

Relation (31) is commonly written as

$$\varepsilon_{\text{eff}}(\varepsilon_1, \varepsilon_2)\varepsilon_{\text{eff}}(\varepsilon_2, \varepsilon_1) = \varepsilon_1\varepsilon_2. \tag{32}$$

Here the first argument of the parameter ε_{eff} denotes the permittivity of the matrix and the second argument denotes the permittivity of the inclusion.

Relations (31) and (32) are equivalent, which follows from the formulas obtained above for the parameter ε_{eff} and also from the analytical expressions for the effective parameters of some two-dimensional systems that admit exact solutions. As an example, we consider a system with chessboard structure, whose black and white crates are represented by different dielectrics with permittivities ε_1 and ε_2. As is known [11, 12], this isotropic heterogeneous system has the effective permittivity

$$\varepsilon_{\text{eff}} = \sqrt{\varepsilon_1\varepsilon_2} \tag{33}$$

and satisfies the reciprocal relation (32).

Introducing the parameter Δ_{12}, we bring formula (33) to the form

$$\varepsilon_{\text{eff}} = \varepsilon_1\sqrt{\frac{1-\Delta_{12}}{1+\Delta_{12}}}. \tag{34}$$

The reciprocal relation takes the form of (31).

If we convert to the expression for the relative effective permittivity Δ_{eff}, formula (34) becomes

$$\Delta_{\text{eff}} = \frac{2}{\Delta_{12}}\Big(1-\sqrt{1-\Delta_{12}^2}\Big). \tag{35}$$

From formula (35) it follows that the reciprocal relation is written as the condition of oddness for the function $\Delta_{\text{eff}}(\Delta_{12})$:

$$\Delta_{\text{eff}}(\Delta_{12}) = -\Delta_{\text{eff}}(\Delta_{21}). \tag{36}$$

A plot of this function is symmetrical about the coordinate origin (dashed curve in Fig. 5).

For the relative effective permittivity for a three-component medium (25), the reciprocal relation is also written as the condition of oddness for the function of two independent variables:

$$\Delta_{\text{eff}}(\Delta_{12}, \Delta_{13}) = -\Delta_{\text{eff}}(\Delta_{21}, \Delta_{31}). \tag{37}$$

This relation follows from expression (25) with allowance for the evenness and oddness of the functions $\Phi^+(\Delta_{12}, \Delta_{13})$ and $\Phi^-(\Delta_{21}, \Delta_{31})$, respectively, which are defined by formulas (27) (see also Fig. 5).

Relation (37) is a generalization of relation (36). Continuing the calculation process, by induction, we obtain the reciprocal relation for isotropic matrix composites with an arbitrary number of components. If, for example, a dielectric material with permittivity ε_1 contains n types of inclusions with permittivities $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_{n+1}$, the reciprocal relation for the relative effective permittivity of this composite is written as

$$\Delta_{\text{eff}}(\Delta_{12}, \Delta_{13}, \ldots, \Delta_{1(n+1)}) = -\Delta_{\text{eff}}(\Delta_{21}, \Delta_{31}, \ldots, \Delta_{(n+1)1}). \tag{38}$$

The reciprocal relation reflects the most general properties of an isotropic matrix material irrespective of its structure and the number and concentrations of components. Statement (38) in the form of a theorem was proved by Keller [13] for a two-dimensional two-component matrix material with cylindrical inclusions (Rayleigh model [10]). For sufficiently general assumptions, the validity of the reciprocal relation for two-dimensional binary systems was proved by Mendelson using algebraic methods [14].

The reciprocal relation is of general-theoretical significance but it is also used in practical calculations to check the accuracy of calculations of effective parameters.

The reciprocal relation (32) can be easily interpreted from a physical viewpoint. For composites containing more than two components this is not so. The reciprocal relations (37) and (38) are also typical for phenomenological coefficients of electromagnetism.

Conclusions. The above system was studied using exact solutions of the model problems of interaction of two dissimilar cylindrical bodies in an external homogeneous electric field. Pair interactions of inclusions are summed up with allowance for the finite range of their mutual influence. Obviously, the wider the influence region, the larger the number of inclusions in the region of interactions and the higher the accuracy in calculations of averaged characteristics of the heterogeneous medium. However, with increase in the influence region, the volume of calculations increases rapidly. In the adopted calculation procedure, the calculation error can be evaluated in each step of solution of the problem.

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